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RESEARCH ARTICLE

Managing the risk of loan prepayments and the optimal structure of short term lending rates

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Abstract Loan pricing is an extremely important aspect of bank operations because loans are typically over two-thirds of bank assets. Many researchers have analyzed the theoretical and empirical impact of how different factors should and do affect fixed rate loan rates and loan prepayments. However, a theoretical decision making model for maximizing expected profit in a declining rate environment has not been developed. After describing the conditions for the optimal loan rate, we develop numerical solutions for it under varying conditions. The varying conditions include the trend in interest rates, volatility of interest rates, and loan maturity.

Keywords Loans · Prepayment · Interest rate risk

JEL Classification Number G21

1 Introduction

Debt prepayment is a very important topic for debt issuers and debt purchasers.[1] Prepayment of various types of *loans* has been empirically analyzed by many researchers.

[1] The value of a corporate bond's call feature has been studied by King (2002), among others.

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A primary focus of empirical prepayment research is the myriad of factors that affect and hopefully help predict prepayment behavior.

Pennington-Cross (2003) analyzes prepayment on prime and nonprime mortgages. He finds that the borrower's credit score, estimated probability of negative equity, level of unemployment, estimated option value of calling the loan, and age of the loan clearly affect prime loan prepayment. Deng et al. (2000) model mortgage prepayment as dependent on estimated option values, loan to value ratios, state unemployment rates, and even state divorce rates. Unlike the two above studies, Heitfield and Sabarwal (2004) empirically analyze auto loans and how unemployment rates, household debt service, and personal bankruptcy cases affect prepayment.[2]

Of course, prepayment is not the only type of risk owners of loan portfolios are exposed to. Default risk is also very important. Using a hazard model, Shumway (2001) has analyzed default and bankruptcy risk for corporations. Gross and Souleles (2002) analyze personal bankruptcy and delinquency using credit card accounts.[3]

Our purpose, in contrast to the above, is not empirical. Instead, our purpose is to abstract from the above empirical analysis and provide a theoretical decision making model for the banker who wishes to manage prepayment risk on fixed rate loans. More specifically, our model recognizes that a banker who sets the fixed loan rate at what the market will bear, without consideration of prepayment risk, exposes the bank to greater (prepayment) risk than if charges a lesser rate. We answer several important questions. Focusing on prepayment risk, what rate should a bank charge on a loan to maximize expected profit? How will this rate be affected by the trend in interest rates, the volatility of interest rates, and the maturity of the loan?[4] Bankers will find our analysis useful for increasing expected profits.

The next section describes the objective function that motivates the search for an optimal interest rate on a fixed rate bank loan. We use a stochastic process to describe the behavior of short term interest rates across time. A benchmark process represents the cost of alternative funding available to the loan recipient throughout the lending period. If the benchmark rate falls below the loan rate, the borrower refinances his loan. As a consequence, the bank is forced to re-lend the money at a rate of interest lower than the original lending rate and suffer the transactions costs associated

[2] The empirical research noted above has recognized heterogeneity across loans. For example, Heitfield and Sabarwal (2004) note that auto loan analysis differs from mortgage loan analysis in that, as time passes, autos do not hold value as well as real estate.

[3] Furthermore, Pennington-Cross (2003); Deng et al. (2000); and Heitfield and Sabarwal (2004) all consider default risk in addition to prepayment risk. With respect to differing credit quality of mortgage loans, Pennington-Cross (2003) finds that nonprime loan prepayments are less responsive to declines in interest rates.

[4] Extraordinary recent advances in modeling economic time series have greatly improved models of forecasted price and yield changes. Cochrane (1999) lists return (price) predictability as one of the new facts of finance and maintains that both equity and bond returns are somewhat predictable. Xu (2004) has found that utilizing small levels of predictability can result in higher returns that are less risky. Dhillon and Lasser (1998) find they can predict future interest rates to a degree. Papageorgiou and Skinner (2002) find that they can predict the direction of interest rate changes. Also, Mills (1999) finds some predictability in UK gilt yields. Pindyck and Rubinfeld (1998) find predictability in US Treasury bond yields as do Jones et al. (1998). Baker et al. (2003) predict bond returns. Finally, Reisman and Zohar (2004) find some price (yield) predictability. Of course, bankers should use any forecasting ability to increase profitability.

with closing the original loan and with opening a new loan. On the other hand, if the benchmark rate remains far above the loan throughout the horizon then the bank could have clearly charged the borrower more for the use of the funds. We describe specification of the prepayment likelihood that depends on, among other things, the forecasted trend in interest rates. In the following section, we derive the expressions needed to solve for the optimal loan rate and provide simulations of optimal loan rates. We include analysis of how maturity, volatility and trends in interest rates interact to determine optimal loan rates. The last section concludes the paper.

2 The model

2.1 Time to prepayment

Our objective function is an inter-temporal expected profit function that begins at t_0, the inception of the loan, and ends at T_M, the time to maturity of the loan. We first derive the likelihood function of the time to prepayment. Then we develop an expression for expected profit. The model has a benchmark rate of interest whose continuous changes across time are given by trended Brownian motion.[5] If the benchmark rate of interest remains above r_L throughout the time horizon, then the fixed loan recipient will not have reason to payoff the loan before maturity and the bank's revenue would be

$$r_L \cdot \mathrm{L} \cdot (T_M - t_0). \tag{1}$$

However, if the benchmark rate falls below r_L at some point in time T_I, then prepayment occurs.[6] Here T_I refers to some intermediate point in time. The bank suffers fixed transaction costs of 2τ and will have to relend the funds at a rate which is lower than r_L, r_I. Thus, revenue from loan L would be[7]

$$[r_L \cdot \mathrm{L} \cdot (T_I - t_0)] - 2\tau + [r_I \cdot \mathrm{L} \cdot (T_M - T_I)]. \tag{2}$$

[5] Although interest rates in general have no trend, we believe it is reasonable to assume that interest rates vary systematically with aggregate economic activity. In particular, interest rates increase over the expansionary phase of the business cycle and fall during contractions. For our model, we assume that R(t) follows a Brownian motion process with drift of μ where $\mu < 0$ because the threat of prepayment only arises in a period of falling interest rates. In order to gain support for characterizing the behavior of short term interest rates as trended Brownian motion, we reviewed domestic interest rates over the last 20 years. We discovered five intervals of time where short term rates declined by at least 200 basis points (see Table 1) and then tested the appropriateness of a linear trend over the five intervals. In particular, the mean squared error (MSE) of linear, quadratic, and exponential trends were computed for 1 year US Treasury rates across the length of each interval. The linear entries of the right hand side variable fit the data best for all five epochs, justifying our use of trended Brownian motion. Similar results hold for 2 year US Treasury rates.

[6] Borrowers may also wish to prepay a loan even though interest rates have not declined in order to remove restrictive covenants.

[7] Here τ is multiplied by two to capture the costs of both closing the old loan and then finding a replacement loan.

Suppose that R_t is our benchmark rate of interest, a representation of the cost of alternative funding available to the loan recipient throughout the lending period. We will assume that R_t is Brownian motion with a drift of μ (where μ is negative). The time to prepayment T of R_t to the point r_L is characterized by $R_T = r_L$ while if $R_t > r_L$, then $t < T$. That is, we let $p(r; t)$ be the probability $R_t = r$ and that the process does not reach the point of prepayment in the time interval $(0, t)$. Consequently,

$$P\{R_t > r_L \text{ for } t < T\} = \int_{r_L}^{\infty} p(r; t)dr = 1 - P(r_L; t) = 1 - \int_{-\infty}^{r_L} p(r; t)dr$$

is not just one minus the cumulative probability density function of R_t but is also the probability that prepayment has not occurred by t. In other words, we have

$$1 - P(r_L; t) = \text{prob}(t < T)$$

as the probability of no prepayment and

$$P(r_L; t) = \text{prob}(T < t)$$

as the probability of prepayment. The cumulative density of T is

$$\int_{\infty}^{r_L} p(r; t)dr = G(t; r_L),$$

and the marginal probability density is

$$\frac{\partial P(r_L; t)}{\partial t} = G'(t; r_L) = g(t; r_L).$$

To solve for $g(t; r_L)$ we simply take advantage of the fact that since $p(r; t)$ is Brownian motion, it satisfies Kolmogorov's diffusion equation. Since $P(r_L; t)$, like $p(r; t)$, satisfies the diffusion equation, then $g(t; r_L)$ must also satisfy Kolmogorov's equation. So we have

$$\frac{\partial g(t;\ r_L)}{\partial t} = \mu \frac{\partial g(t;\ r_L)}{\partial r_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 g(t; r_L)}{\partial r_0^2}$$

where r_0 is the initial rate of interest of our Brownian motion and σ is a parameter of R_t's standard deviation.

The expression above is a partial differential equation whose direct solution is quite difficult. It is far more convenient to solve for $g(t; r_L)$ using Laplace transforms with respect to time. The details of the Laplace transformation and its solution will be provided by the authors upon request. The analysis yields the probability density of the time to prepayment given below as

$$\mathrm{g(t)} = \frac{|r_{\mathrm{L}} - r_0|}{\sigma\sqrt{2\pi t^3}}\,\mathrm{e}^{-(r_{\mathrm{L}} - r_0 - \mu t)^2/2\sigma^2 t}.$$

2.2 The expected profit function

Recall that if the time to prepayment is greater than the time to maturity of the loan, then the contribution of the loan to total revenue is $r_{\mathrm{L}} \cdot \mathrm{L} \cdot (T_{\mathrm{M}} - t_0)$. The present value contribution of this possibility to expected profit, $E(\pi)$, is thus

$$\mathrm{e}^{-\lambda \mathrm{TM}} \cdot r_{\mathrm{L}} \cdot \mathrm{L} \cdot T_{\mathrm{M}} \cdot \int_{T_{\mathrm{M}}}^{\infty} \mathrm{g(t)d}t \tag{3}$$

where $t_0 = 0$, λ is the bank's cost of capital, and $\int_{T_{\mathrm{M}}}^{\infty} \mathrm{g(t)}\,\mathrm{d}t$ is the likelihood that the loan reaches maturity.[8] If prepayment occurs, then the fixed costs of processing both loan prepayment and "re-issuing" a replacement loan are incurred (2τ). Their discounted inclusion into $E(\pi)$ is written as

$$2\tau \int_{0}^{T_{\mathrm{M}}} \mathrm{e}^{-\lambda t}\mathrm{g(t)d}t \tag{4}$$

where $\int_{0}^{T_{\mathrm{M}}} \mathrm{g(t)}\,\mathrm{d}t$ is the likelihood of prepayment over the life of the loan.

If the prepayment occurs, then the revenue from loans across the time horizon to T_{M} would be

$$r_{\mathrm{L}} \cdot \mathrm{L} \cdot T_{\mathrm{I}} + r_{\mathrm{I}} \cdot \mathrm{L} \cdot (T_{\mathrm{M}} - T_{\mathrm{I}}). \tag{5}$$

T_{I} is an intermediate point in time where $t_0 < T_{\mathrm{I}} < T_{\mathrm{M}}$. In other words, the time to prepayment is given by the random variate T and if prepayment takes place before T_{M} we denote it as T_{I}. When weighted by the likelihood of occurrence and discounted to present value, the first term in Eq. (5) becomes simply

$$r_{\mathrm{L}} \cdot \mathrm{L} \cdot \int_{0}^{T_{\mathrm{M}}} \mathrm{te}^{-\lambda t}\,\mathrm{g(t)}\,\mathrm{d}t. \tag{6}$$

[8] Using current market date from Bloomberg on the cost of equity, debt, and preferred stock for a large sample of publicly traded banks in the United States, we find a weighted average cost of capital (WACC) of approximately 5%. We then used this WACC as the bank's inter-temporal discount rate, λ.

The evaluation of the discounted expected value of the second term in equation (5) involves two random variables and yields

$$\mathrm{L}\int_{0}^{T_{\mathrm{M}}}(T_{\mathrm{M}}-t)\left[\int_{t}^{T_{\mathrm{M}}}\left[\int_{-\infty}^{\infty} r\frac{1}{\sqrt{2\pi\sigma^{2}\Delta}}\times \mathrm{e}^{-(r-r_{\mathrm{L}}-\mu\Delta)^{2}/2\sigma^{2}\Delta}\,\mathrm{d}r\right]\mathrm{e}^{-\lambda\Delta}\left(\frac{1}{T_{\mathrm{M}}-t}\right)\mathrm{d}\Delta\right]\mathrm{g(t)}\,\mathrm{d}t \tag{7}$$

The innermost integral involves our benchmark rate that continues to evolve through time but since prepayment has occurred, its dynamic behavior is over the period that we now call Δ. At any Δ_0 the probability density function of r is given by

$$\frac{1}{\sqrt{2\pi\sigma^{2}\Delta_{0}}}\,\mathrm{e}_{0}^{-(\mathrm{r}-\mathrm{r}_{\mathrm{L}}-\mu\Delta_{0})^{2}/2\sigma^{2}\Delta_{0}}.$$

Note that r begins at r_{L} for $\Delta = 0$ to acknowledge that prepayment takes place at r_{L} which is the incipiency of Δ. The expression $\int_{-\infty}^{\infty} r\frac{1}{\sqrt{2\pi\sigma^{2}\Delta_{0}}}\,\mathrm{e}^{-(r-r_{\mathrm{L}}-\mu\Delta_{0})^{2}/2\sigma^{2}\Delta_{0}}\,\mathrm{d}r$ is a conditional mean, the mean of our trended Brownian motion at Δ_0. Integrating with respect to Δ over the limits of t to T_{M} provides an accumulation of all the rates of interest expected to exist from the time of prepayment until the original loan was to mature. Multiplying the integrand by $1/(T-t)$, yields a mean rate of interest that is expected to exist over the period t to T_{M}. The product of $(T_{\mathrm{M}}-t)$ and the bracketed double integral is weighted by g(t) and then integrated over all epochs to yield the expected return on the replacement loan.[9] Finally, multiplying this expression by L results in the present value of the expected level of income from the replacement loans. Collecting (3), (4), (5), (6) and (7) yields the expected profit function which is

$$\begin{aligned}E(\pi) = {} & T_{\mathrm{M}}\mathrm{L}r_{\mathrm{L}}\mathrm{e}^{-\lambda T_{\mathrm{M}}}\int_{T_{\mathrm{M}}}^{\infty}\mathrm{g(t)d}t + r_{\mathrm{L}}\mathrm{L}\int_{0}^{T_{\mathrm{M}}} t\mathrm{e}^{-\lambda t}g(t)\mathrm{d}t - T_{\mathrm{M}}\,\mathrm{D}r_{\mathrm{D}}\mathrm{e}^{-\lambda T_{\mathrm{M}}}\\ & - r_{\theta}(\mathrm{L}-\mathrm{D}-\mathrm{K})T_{\mathrm{M}}\mathrm{e}^{-\lambda T_{\mathrm{M}}} + \mathrm{L}\int_{0}^{T_{\mathrm{M}}}(T_{\mathrm{M}}-t)\left[\int_{t}^{T_{\mathrm{M}}}\left[\int_{-\infty}^{\infty} r\frac{1}{\sqrt{2\pi\sigma^{2}\Delta}}\right.\right.\\ & \left.\left.\times\mathrm{e}^{-(r-r_{\mathrm{L}}-\mu\Delta)^{2}/2\sigma^{2}\Delta}\,\mathrm{d}r\right]\mathrm{e}^{-\lambda\Delta}\frac{1}{T_{\mathrm{M}}-t}\,d\Delta\right]\mathrm{g(t)}\,\mathrm{d}t - 2\tau\int_{0}^{T_{\mathrm{M}}}\mathrm{e}^{-\lambda t}\mathrm{g(t)d}t\end{aligned} \tag{8}$$

where r_{D} is the cost of deposits, K is bank capital, D is bank deposits, and r_θ is the cost of bank borrowing if deposits and capital are not adequate.

[9] In order to enhance this explanation, a simple discrete example is available from the authors upon request.

The present value of deposit funding (D) is given by

$$e^{-\lambda T_M} \cdot r_D \cdot D \cdot T_M$$

while emergency funding of the loan is provided at a present value cost of

$$e^{-\lambda T_M} \cdot r_\theta \cdot (L - D - K) \cdot T_M.$$

when borrowed funds are used to make up a discrepancy between the uses and sources of bank funds.

Scrutinizing equation (8), it is clear that our output price has many points of incidence in our expected profit function. The decision variable is the amount earned on the original level of loans provided. However, the output price is also embedded in the probability density function of the time to prepayment g(t). Of course, g(t) is the basis for the likelihood of, and the mean time to, prepayment. In order to help understand the tradeoffs involved in maximizing (8), consider an increase in the lending rate: the greater the r_L, the greater the earnings on the original loan. However, if the lending rate increases, the mean time to prepayment, $\int_0^{T_M} tg(t)dt$, falls. Although the increase in r_L will increase the amount earned on the original bank loan it will shorten the expected longevity of the loan. In addition, when the prepaid funds are re-lent (within the horizon) the expected return will be less than r_L since μ is negative. It is also the case that an increase in the lending rate increases the likelihood of prepayment and, consequently, increases the probability that the bank will suffer the fixed costs of loan processing, 2τ. Clearly, the concavity of the expected profit function emanates from the ability of r_L to balance these expected revenue and expected cost considerations.

2.3 Optimal short term lending rate

Maximizing our inter-temporal expected profit function with respect to r_L yields

$$\begin{aligned}
&T_M L e^{-\lambda T_M} \int_{T_M}^{\infty} g(t)dt + T_M r_L \frac{\partial L}{\partial r_L} e^{-\lambda T_M} \int_{T_M}^{\infty} g(t)dt + T_M L r_L e^{-\lambda T_M} \left[\frac{\partial}{\partial r_L} \int_{T_M}^{\infty} g(t)dt \right] \\
&+ L \int_0^{T_M} te^{-\lambda t} g(t)dt + r_L \frac{\partial L}{\partial r_L} \int_0^{T_M} te^{-\lambda t} g(t)dt + r_L L \frac{\partial}{\partial r_L} \left[\int_0^{T_M} te^{-\lambda t} g(t)dt \right] - r_\theta \frac{\partial L}{\partial r_L} T_M e^{-\lambda T_M} \\
&+ \left[\frac{\partial L}{\partial r_L} + L \frac{\partial}{\partial r_L} \right] \left\{ \int_0^{T_M} (T_M - t) \left[\int_t^{T_M} \left[\int_{-\infty}^{\infty} r \frac{1}{\sqrt{2\pi\sigma^2 \Delta}} e^{-(r - r_L - \mu\Delta)^2 / 2\sigma^2 \Delta} dr \right] \right. \right. \\
&\left. \left. \times e^{-\lambda\Delta} \left(\frac{1}{T_M - t} \right) d\Delta \right] g(t)\, dt \right\} - 2\tau \frac{\partial}{\partial r_L} \left[\int_0^{T_M} e^{-\lambda t} g(t)dt \right] = 0.
\end{aligned} \tag{9}$$

The integral differential equation above refuses to yield an explicit solution for the optimal loan rate. Examining Eq. (9), it is clear that the decision variable $\mathrm{r_L}$ affects the first order condition (FOC) in a number of ways. That is, r_{L} appears in the limit of three integrals, in twelve integrands, as the power to which nearly every exponential function is raised, and in the determination of the slope of the loan demand schedule. Given the disparate appearance of the optimal output price in Eq. (9), normalizing on r_{L} is impossible. Not being able to isolate r_{L} on the left hand side of such a complex FOC mutes any immediate intuition. In addition, the complexity of the FOC nearly guarantees that using the implicit function theorem for comparative static analysis will be fruitless. However, simulating the bank's optimal loan rate and performing a comparative static analysis upon r_{L}^* will provide us with a good deal of insight.

3 Simulations of the optimal loan rate

In regard to preparing Eq. (9) for simulation, we first chose a loan demand equation which is widely used in the literature,

$$\mathrm{L} = \mathrm{e}^{-\alpha\, r_{\mathrm{L}}}.$$

The exponential function above yields an inverse relationship between L and r_{L} suggesting that an increase in the lending rate decreases the quantity of bank loans demanded. The elasticity of demand for bank loans is given by

$$\frac{\partial L}{\partial r_{\mathrm{L}}}\frac{r_{\mathrm{L}}}{\mathrm{L}} = -\alpha r_{\mathrm{L}}$$

so that

$$\frac{\partial L}{\partial r_{\mathrm{L}}}\frac{1}{\mathrm{L}} = -\alpha.$$

Dividing the LHS and the RHS of Eq. (9) by the level of loans (L), we can replace the term $\frac{\partial L}{\partial r_{\mathrm{L}}}\frac{1}{\mathrm{L}}$ with $-\alpha$ wherever it appears.[10]

Examining the expected profit function further, it is clear that we still need to specify μ, σ and T_{M}. In order to employ realistic characterizations of the parameters μ and σ, we reviewed the behavior of US interest rates over the last 25 years. We discovered five intervals of time where short term interest rates, represented by 1 year US Treasury rates, suffered a sustained decrease in the rate of interest of at least 200 basis points. Table 1 details the fall in the yields over time on 1 year treasury securities, the estimated μ, and the estimated σ. Notes on the procedures used to estimate the parameters of trended Brownian motion will be provided upon request.

[10] To ensure a finite optimality condition for r_{L}, we assume that the elasticity of demand is less than one.

Table 1 Episodes of sustained declines in interest rates

Time period	Number of months	Change in basis points	Annualized μ	Annualized σ
June 29, 1984 –October 10, 1986	30	−671	−0.02908	0.01093
March 31, 1989 –December 22, 1989	10	−205	−0.02733	0.01095
May 4, 1990 –October 9, 1992	32	−547	−0.02222	0.00738
January 6, 1995 –February 23, 1996	15	−220	−0.01907	0.00599
November 24, 2000 –June 27, 2003	34	−507	−0.01939	0.00774

The integral differential equation given by (9) was solved numerically using Newton's search technique for first order conditions.[11] Figures 1a and b depict the behavior of expected profits for maturities of one and 2 years, respectively. Expected profits clearly behave such that there is a maximum value. That is, the expected profit function clearly peaks at r_{L}^{*} of 7.43% for a loan that matures in 1 year (Fig. 1a) while, for the 2 year loan, $T_{\mathrm{M}} = 2$, the optimal lending rate is 5.83% (Fig. 1b).[12] The fact that r_{L}^{*} is declining with respect to maturity is interesting. An increase in T_{M} impacts all six right hand side terms in (8) and, given its disparate entry, T_{M}'s net impact on expected profit is impossible to sign analytically. However, insight into the inverse relationship between T_{M} and r_{L}^{*} can be gained by considering the risk of loan prepayment. The risk of prepayment, or more accurately the likelihood of prepayment, is given by $\int_0^{T_{\mathrm{M}}} g\,(t; r_{\mathrm{L}}^{*})\,\mathrm{d}t$ and appears in four of the right hand side terms of expected profit. While $g(t;\ r_{\mathrm{L}}^{*})$ is weighted in three of the four cases, understanding how $\int_0^{T_{\mathrm{M}}} g\,(t; r_{\mathrm{L}}^{*})\,\mathrm{d}t$ behaves in the face of parametric changes is useful in understanding how our inter-temporal expected profit behaves in the face of the same changes.

[11] The details associated with using Newton's search technique for finding r_{L}^{*} will be provided by the authors upon request. Our base case parameters were T_{m} (1.5), μ (−0.01907), α (1.0), σ (0.00848), r_{D} (0.02), r_0 (0.0873), r_θ (0.015), τ (0.02), Δ (1/730), λ (0.05), Φ (0.075), and γ (0). The value of μ is drawn from an epoch in Table 1.

[12] Such a term structure is consistent with conditions where interest rates are expected to decline. The expectations theory of term structure suggests that long term rates are an average of expected short term rates. An environment where rates are expected to decline explains a negative term structure that, for example, frequently accompanies time periods just before recessions and the early stages of a recession. Our analysis suggests an additional reason for the term structure of bank loans to be negative. Borrowers are more likely to exercise their prepayment rights so that banks should charge less. We also note that the reader may be tempted to relate our analysis to the impact of a call feature upon callable bond yields. However, such comparisons do not yield parallel intuition or results. For example, a greater σ means greater option value and greater required yield for a callable bond. In our case, a greater σ leads to a lower interest rate. This is because our model has a demand curve for loans.

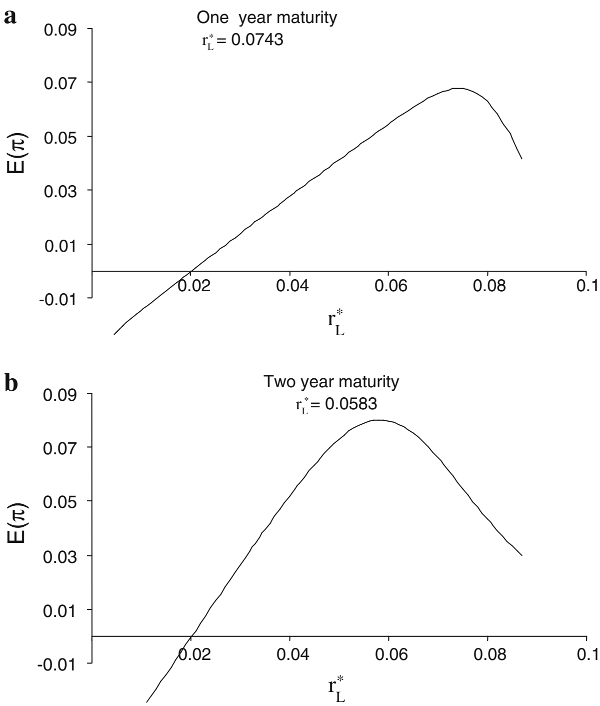

Fig. 1 **a** Values used are T_m (1.0), μ (−0.02315), α (1.0), σ (0.00848), r_D (0.02), r_0 (0.0873), r_θ (0.015), τ (0.02), Δ (1/730), λ (0.05), and γ (0). **b** Values used are T_m (2.0), μ (−0.02315), α (1.0), σ (0.00848), r_D (0.02), r_0 (0.0873), r_θ (0.015), τ (0.02), Δ (1/730), λ (0.05), γ (0)

For example, the partial derivative of the risk of prepayment with respect to T_M is easily found to be $\left[\frac{|r_L^*-r_0|}{\sigma\sqrt{2\pi\, T_M^3}} e^{\frac{-(r_L^*-r_0-\mu T_M)^2}{2\sigma^2 T_M}}\right]$, a positive number. Ceteris paribus, the longer the horizon, T_M, the greater the chance of loan prepayment because each additional moment in time provides the benchmark rate another opportunity to fall below r_L^*. The partial derivative of $\int_0^{T_M} g\,(t;\, r_L^*)\,\mathrm{d}t$ with respect to r_L^* is $\int_0^{T_M} g\,(t;\, r_L^*)$ $\left\{\frac{\mu t + r_0 - r_L^*}{\sigma^2\, t} - \frac{1}{|\, r_L^* - r_0\,|}\right\} \mathrm{d}t$, a positive number for the parameters used in our simulations. If preserving the existing risk of prepayment were the bank's objective function, then the implicit function theorem would suggest that r_L^* will fall with an increase in T_M.

$$\frac{\mathrm{d}r_L^*}{\mathrm{d}\,T_M} = (-1)\,\frac{\dfrac{\partial \int_0^{T_M} \mathrm{g}\,(\mathrm{t}; r_L^*)\,\mathrm{d}t}{\partial T_M}}{\dfrac{\int_0^{T_M} \mathrm{g}\,(\mathrm{t}; r_L^*)\,\mathrm{d}t}{\partial r_L^*}} < 0$$

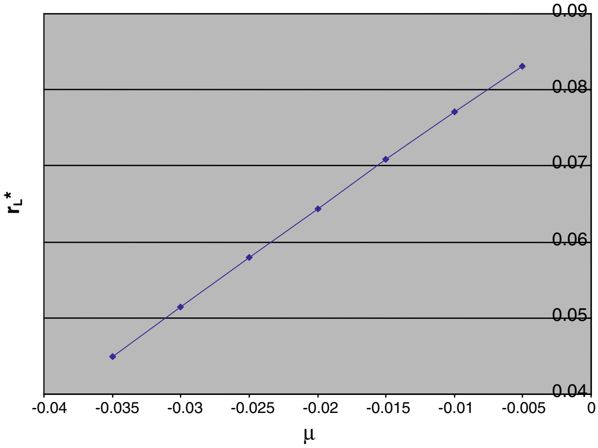

Fig. 2 Values used are T_M (1.5), μ (−0.01907), α (1.0), σ (0.00848), r_D (0.02), r_0 (0.0873), r_θ (0.015), τ (0.02), Δ (1/730), λ (0.05), Φ (0.075) and γ (0)

This kind of intuition can be extended to Figs. 1a and b. Increasing T_M increases the risk of loan prepayment. This increase impacts expected profits and, consequently, r_L^* falls to offset the impact of the increased likelihood of loans being prepaid upon expected profits.

Examining Fig. 2, it is clear that increases in μ increase the optimal rate of lending (presuming, of course, that μ remains below zero). Rather than examining expected profit directly to explain the behavior of r_L^*, again intuition is well served by considering the risk of loan prepayment. The partial derivative of $\int_0^{T_M} g\,(t; r_L^*)\,\mathrm{d}t$ with respect to μ is given by

$$\int_0^{T_M} \left[\frac{|r_L^* - r_0|}{\sigma\sqrt{2\pi\, t^3}} \mathrm{e}^{\frac{-(r_L^* - r_0 - \mu\, t)^2}{2\sigma^2 t}} \right] \left[\frac{r_L^* - r_0 - \mu\, t}{\sigma^2} \right] \mathrm{d}t$$

which is negative. While $\partial \int_0^{T_M} g\,(t; r_L^*)\,\mathrm{d}t \,/\partial\, r_L^*$ is positive, the implicit function theorem yields $(\mathrm{d}\, r_L^* / \mathrm{d}\mu) > 0$ for $\mathrm{d}\left[\int_0^{T_M} g\,(t; r_L^*)\,\mathrm{d}t\right] = 0$. An increase in μ decreases the risk of prepayment and, consequently, the bank is able to increase r_L^* without increasing the optimal risk of loan prepayment.

The partial derivative of $\int_0^{T_M} g\,(t; r_L^*)\,\mathrm{d}t$ with respect to σ is

$$\int_0^{T_M} g(t; r_L^*) \left[\frac{(r_L^* - r_0 - \mu\, t)^2}{\sigma^3\, t} - \frac{1}{\sigma} \right] \mathrm{d}t,$$

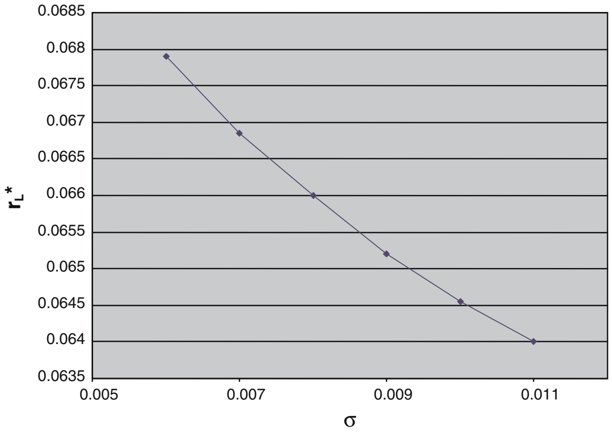

Fig. 3 The relation of the optimal lending rate to σ. The optimal rate declines with σ

a positive number for all realistic values of σ, so that an increase in σ increases the risk of loans being paid off before their scheduled maturity. With $\partial\left[\int_0^{T_M} g\,(t;\,r_L^*)\mathrm{d}t\right]/\partial\,r_L^* > 0$, then

$$\frac{\mathrm{d}\,r_L^*}{\mathrm{d}\,\sigma} = (-1)\,\frac{\dfrac{\partial\int_0^{T_M} g\,(t;r_L^*)\,\mathrm{d}t}{\partial\sigma}}{\dfrac{\partial\int_0^{T_M} g\,(t;r_L^*)\,\mathrm{d}t}{\partial r_L^*}} < 0.$$

An exogenous increase in σ occasions a decrease in r_L^* if the existing risk of prepayment is to be maintained. Extending this logic to explain the results in Fig. 3 leads us to think an increase in σ increases the role of $\int_0^{T_M} g\,(t;\,r_L^*)\,\mathrm{d}t$ in Eq. (8) and this occasions a reduction in r_L^* as depicted.

Next, consider the impact of loan maturity, T_M as given in Fig. 4. Here r_L^* declines with maturity where the reason is that the longer the maturity, the more likely interest rates will fall thus causing prepayment. Thus, the longer the maturity, the lower the rate bankers should charge to maximize expected profit. This suggests a negative term structure of loan rates.

Such a term structure is consistent with conditions where interest rates are expected to decline. The expectations theory of term structure suggests that long term rates are an average of expected short term rates. An environment where rates are expected to decline explains a negative term structure that, for example, frequently accompanies time periods just before recessions and the early stages of a recession. Our analysis suggests an additional reason for the term structure of bank loans to be negative.

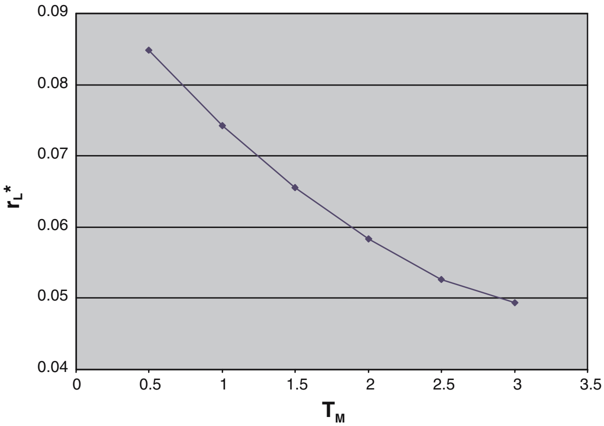

Fig. 4 The relation of the optimal lending rate to maturity. The optimal rate declines with maturity because, as maturity increases, it is more likely interest rates will eventually fall far enough to cause prepayment

Borrowers are more likely to exercise their prepayment rights so that banks should charge less.[13]

4 Impact of parameter interaction on optimal loan rates

While greater maturity, interest rate volatility, and negative interest rate trends all reduce the optimal rate to charge on a loan, it is also interesting and useful to analyze how these factors interact with each other to impact r_{L}^*. In some cases the deterioration in r_{L}^* may be enhanced by interaction with another parameter while in other cases the impact may be practically invariant. For these simulations we use an assumed μ of -0.005, a milder deterioration in interest rates than above. We do this because interest rates may decline much more slowly than the severe declines noted above.

Figure 5 depicts r_{L}^* as it varies with μ where the different lines represent varying maturities. The spread between a 1 year and 2 years grows with increasingly negative μ. That is, for μ of -0.004 the spread between the 1 year and 2 year is 53 basis points while the spread grows to 94 basis points at a μ of -0.010. The apparent explanation is that as the interest trend becomes more negative, the likelihood of prepayment accelerates much more rapidly if maturity also increases. Of course, prepayment incurs undesirable fixed costs of finding a borrower and lower asset yield which the bank tries to avoid by reducing r_{L}^*. Figure 6 also depicts r_{L}^* as it varies with μ but the different lines represent different σ. The bank should charge a higher rate for lower σ values but the spread slightly declines with more negative μ's.[14] The apparent explanation

[13] The reader may be tempted to relate our analysis to the impact of a call feature upon callable bond yields. However, such comparisons do not yield parallel intuition or results. For example, a greater σ means greater option value and greater required yield for a callable bond. In our case, a greater σ leads to a lower interest rate. This is because our model has a demand curve for loans.

[14] At μ of -0.004, the spread between σ of 0.007 and 0.010 is 41 basis points. At μ of -0.01, the spread is 35 basis points.

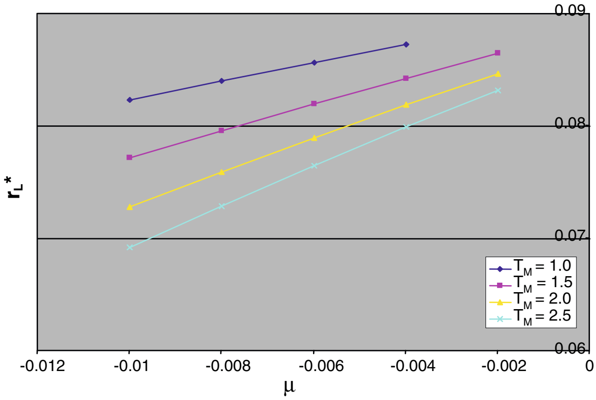

Fig. 5 The relation of optimal loan rate to μ for different maturities

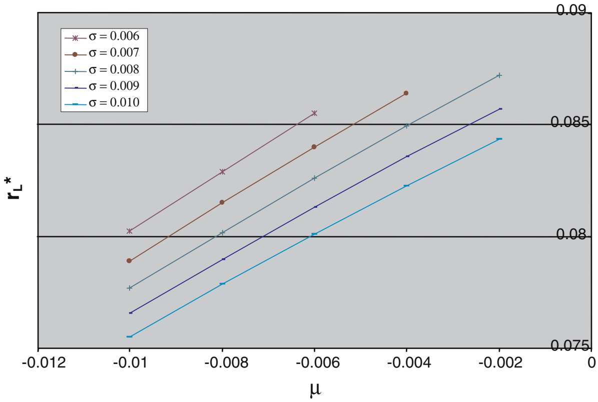

Fig. 6 The relation of optimal loan rate to μ for different σ values

of the change in spread is that stronger negative trends in interest rates dominate optimal solutions so that volatility makes less difference. In other words, if μ is strong, prepayments will not be much affected by volatility.

Figure 7 depicts r_L^* as it varies with maturity where the different lines represent different μ's. As maturity increases, the spread due to μ increases. At a 1 year maturity, the spread for a μ of -0.004 is 49 basis points and it is 90 basis points for a μ of -0.010. That is, a stronger μ reduces r_L^* more for a longer maturity apparently because the likelihood of prepayment increases dramatically. Figure 8 depicts r_L^* as it varies with maturity where the different lines represent varying σ's. Here the bank should charge more the lower the σ but this spread due to σ shrinks slightly with greater maturity.[15] The apparent explanation is that as maturity increases, parameters such as μ are more influential than σ on the expected time to prepayment.

[15] At a maturity of 1 year the spread between σ of 0.007 and 0.010 is 27 basis points. At a maturity of 2 years, the spread is 23 basis points.

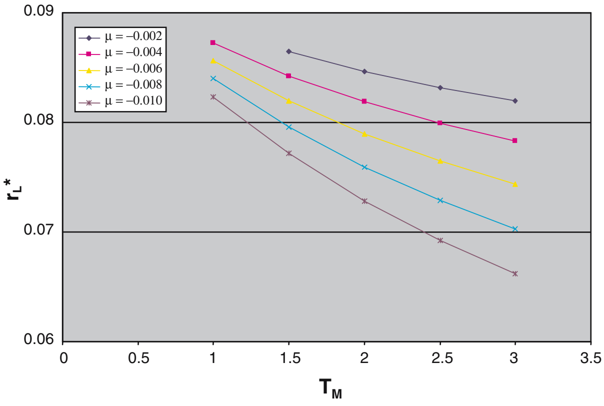

Fig. 7 The relation of optimal loan rate to T_M for different μ values

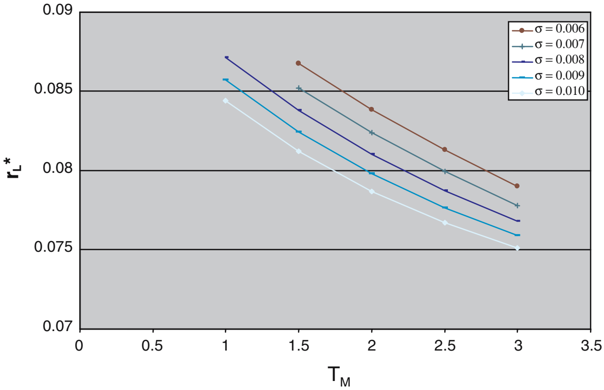

Fig. 8 The relation of optimal loan rate to T_M for different σ values

Figure 9 depicts r_L^* as it varies with σ where the different lines represent varying maturities. Here the spread between the 1 year and 2 year is not very sensitive to σ.[16] The interaction between σ and maturity does not accelerate prepayments very much. A figure showing the relation between r_L^* and σ for varying μ's, not shown here but available from the authors, shows a very similar relation where the spread does not change much with σ.

5 The impact of prepayment penalties on loan pricing

In pricing their output banks can attach a penalty (γ) for the repayment of loans before their scheduled maturity. The magnitude of the penalty will clearly impact the optimal lending rate since the benchmark rate of interest must now fall below ($r_L - \gamma$) to

[16] Comparing one versus 2 year maturities, the spread is 61 basis points at a σ of 0.008 and 57 basis points at a σ of 0.01.

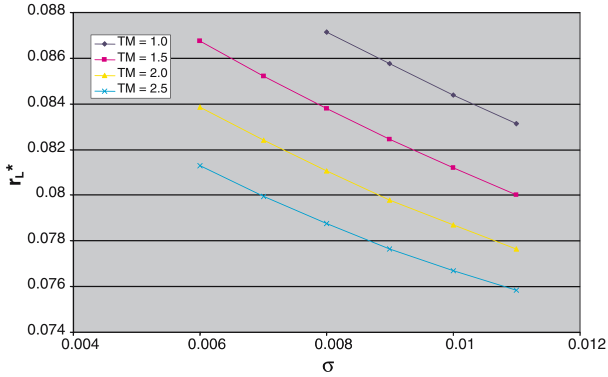

Fig. 9 The relation of optimal loan rate to σ for different T_M values

make the refinancing beneficial. The revised probability density function of the time to prepayment is given by

$$\mathrm{g(t)} = \frac{|r_L - \gamma - r_0|}{\sigma\sqrt{2\pi t^3}} e^{-[(r_L-\gamma)-r_0-\mu t]^2/2\sigma^2 t}$$

where $\mu < 0$ and the conditional density of the benchmark rate after the time of prepayment but before maturity is

$$\frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-[r-(r_L-\gamma)-\mu\Delta]^2/2\sigma^2\Delta}.$$

The new discounted expected profit function is distinguished from (8) in predictable ways and is given as

$$\begin{aligned} \mathrm{E}(\pi) = {} & T_M r_L \mathrm{L} e^{-\lambda T_M} \int_{T_M}^{\infty} \mathrm{g(t)d}t + (r_L+\gamma)\,\mathrm{L}\int_0^{T_M} \mathrm{t}e^{-\lambda t}\mathrm{g(t)d}t - T_M r_D \mathrm{D} e^{-\lambda T_M} \\ & - r_\theta(\mathrm{L}-\mathrm{D}-\mathrm{K})T_M e^{-\lambda T_M} + \mathrm{L}\int_0^{T_M}(T_M - t)\left[\int_t^{T_M}\left[\int_{-\infty}^{\infty} r\frac{1}{\sqrt{2\pi\sigma^2\Delta}}\right.\right. \\ & \left.\left.\times e^{-[r-(r_L-\gamma)-\mu\Delta]^2/2\sigma^2\Delta}\,\mathrm{d}r\right] e^{-\lambda\Delta}\frac{1}{T_M - t}\,\mathrm{d}\Delta\right]\mathrm{g(t)\,d}t - 2\tau\int_0^{T_M} e^{-\lambda t}\mathrm{g(t)d}t \end{aligned} \tag{10}$$

The lending rate r_L appears in the expected profit equation eleven times while γ, the prepayment penalty, enters the expression ten times. In nine instances, the two parameters enter the expected inter-temporal profit function symmetrically but in opposite directions. If (10) is maximized with respect to the lending rate, then the first order condition for r_L^* contains the likelihood of prepayment and the expected time to prepayment nine times. The magnitudes of the derivative of each of these terms with respect to r_L^* and γ are equal but the derivatives themselves are in opposite directions. In particular, we have

$$\left[\frac{\partial \int_0^{T_M} g(t; r_L^*)\,\mathrm{d}t}{\partial r_L^*}\right] = (-1)\left[\frac{\partial \int_0^{T_M} g(t; r_L^*)\,\mathrm{d}t}{\partial \gamma}\right]$$

$$\left[\frac{\partial \int_0^{T_M} t\,g(t; r_L^*)\,\mathrm{d}t}{\partial r_L^*}\right] = (-1)\left[\frac{\partial \int_0^{T_M} t\,g(t; r_L^*)\,\mathrm{d}t}{\partial \gamma}\right]$$

Furthermore, five first derivative expressions that appear in FOC (r_L^*) have second derivatives with respect to r_L^* and γ which are also "equal but opposite". These mathematical characteristics serve to make $\partial\,\mathrm{FOC}\,(r_L^*)\,/\,\partial r_L^*$ very similar in magnitude to $\partial\,\mathrm{FOC}\,(r_L^*)\,/\,\partial\gamma$ in the computation of

$$\frac{\mathrm{d}r_L^*}{\mathrm{d}\gamma} = (-1)\,\frac{\frac{\partial\,\mathrm{FOC}\,(r_L^*)}{\partial\,\gamma}}{\frac{\partial\,\mathrm{FOC}\,(r_L^*)}{\partial\,r_L^*}} \tag{11}$$

In the context of the FOC (r_L^*), when the derivative of the loan demand function is taken with respect to r_L^* and then γ, the results are

$$\frac{\partial \mathrm{L}}{\partial r_L^*} = -\alpha_1\,\mathrm{L} \quad \text{and} \quad \frac{\partial \mathrm{L}}{\partial \gamma} = -\alpha_2\,\mathrm{L}, \quad \text{respectively.}$$

Although both the lending rate and the prepayment penalty impact loan demand inversely, their respective effects are distinctive. It is also the case that $\partial^2\,\mathrm{L}/\partial r_L^*\,\partial r_L^*$ does not equal $\partial^2\mathrm{L}/\partial r_L^*\,\partial\gamma$ which helps to distinguish the magnitude of $\partial\,\mathrm{FOC}\,(r_L^*)\,\big/\partial r_L^*$ from $\partial\,\mathrm{FOC}\,(r_L^*)\,/\,\partial\gamma$ in the determination of (11).

The similar impact that r_L^* and γ have upon $g\,(t;\,r_L^*)$,and its auxiliary expressions, conspire to drive the magnitudes of $\partial\,\mathrm{FOC}\,(r_L^*)\big/\partial r_L^*$ and $\partial\,\mathrm{FOC}\,(r_L^*)\,/\,\partial\gamma$ into correspondence while the distinctive impact that r_L^* and γ have upon loans demanded serve to distinguish the magnitude of $\partial\,\mathrm{FOC}\,(r_L^*)\big/\partial r_L^*$ from that of $\partial\,\mathrm{FOC}\,(r_L^*)/\partial\gamma$. The net impact of the opposing effects was established by the numerical evaluation of the right hand side of equation (11). The simulations used the base values detailed in Sect. 3 and were computed for T_M values of 0.5, 1.0, 1.5 and 2.0 years. For loans with relatively short terms to maturity, T_M = 0.5 and 1.0, the simulations yielded a number that could easily be rounded to one ($dr_L^*\,/\,d\gamma\; = \;$ 1.01 and 1.04, respectively). These results suggest that the similar impact that r_L^* and γ have upon

$g\,(t;\ r^*_{\mathrm{L}})$, and its associated expressions, dominated the calculations. At $T_{\mathrm{M}} = 1.5$ and 2.0 years, loan demand effects $\left(\partial L\,/\,\partial r^*_{\mathrm{L}} \neq \partial L\,/\,\partial\gamma\right)$, meaningfully impact the simulations to the extent that $(\partial r^*_{\mathrm{L}}/\partial\gamma)$ increases to 1.09 and 1.16, respectively.

Characterizing $(\partial r^*_{\mathrm{L}}/\partial\gamma)$ numerically is an artifact of a host of assumptions about the functional form of the loan demand equation, the magnitude of the parameters in the expression for loan demand, the size of τ, $g\,(t;\ r^*_{\mathrm{L}})$'s statistical characteristics, and the time to maturity. In an effort to establish an outcome that may be a little less encumbered and yet speak to the fundamentals at hand, let us consider $(\partial r^*_{\mathrm{L}}/\partial\gamma)$ in a narrower context. Given our preoccupation with the probability density of the time to prepayment, it seems natural to detail the behavior of output prices necessary to maintain the optimal risk of prepayment. In particular, we used the $FOC\ (r^*_{\mathrm{L}})$ obtained from (1A) to compute the optimal lending rate which was substituted into the likelihood of loans being prepaid to yield the optimal risk of prepayment $\int_0^{T_{\mathrm{M}}} g(t;r^*_{\mathrm{L}})\,\mathrm{d}t$.

We then examined just how r^*_{L} changes in order to preserve the optimal likelihood of loans being paid early after a change in γ.

$$\frac{\mathrm{d}r^*_{\mathrm{L}}}{\mathrm{d}\gamma} = (-1)\,\frac{\dfrac{\partial\left[\int_0^{T_{\mathrm{M}}} g(t;r^*_{\mathrm{L}})\,\mathrm{d}t\right]}{\partial\gamma}}{\dfrac{\partial\left[\int_0^{T_{\mathrm{M}}} g(t;r^*_{\mathrm{L}})\,\mathrm{d}t\right]}{\partial r^*_{\mathrm{L}}}}$$

from setting the differential change in the likelihood of prepayment equal to zero.

$$d\left[\int_0^{T_{\mathrm{M}}} g(t;\ r^*_{\mathrm{L}})\,\mathrm{d}t\right] = 0 = \frac{\partial\left[\int_0^{T_{\mathrm{M}}} g(t;\ r^*_{\mathrm{L}})\,\mathrm{d}t\right]}{\partial r^*_{\mathrm{L}}}\,dr^*_{\mathrm{L}} + \frac{\partial\left[\int_0^{T_{\mathrm{M}}} g(t;\ r^*_{\mathrm{L}})\,\mathrm{d}t\right]}{\partial\gamma}\,\mathrm{d}\gamma.$$

Inspecting $g\,(t;\ r^*_{\mathrm{L}})$ it is clear that the lending rate and the prepayment penalty jointly determine the extent to which R(t) must fall before the threat of prepayment is realized. The parameters enter $\int_0^{T_{\mathrm{M}}} g(t;r^*_{\mathrm{L}})\,\mathrm{d}t$ symmetrically but in opposite directions. Consequently, the <u>magnitudes</u> of the partial derivative of the optimal risk of prepayment with respect to r^*_{L} and γ are identical while $\left[\partial\int_0^{T_{\mathrm{M}}} g(t;r^*_{\mathrm{L}})\,\mathrm{d}t/\partial\gamma\right]$ is equal to $(-1)\left[\partial\int_0^{T_{\mathrm{M}}} g(t;r^*_{\mathrm{L}})\,\mathrm{d}t/\partial r^*_{\mathrm{L}}\right]$ so that $(\mathrm{d}\,r^*_{\mathrm{L}}/\mathrm{d}\,\gamma)$ is equal to one.

Of course, the simplicity of the $(\mathrm{d}\,r^*_{\mathrm{L}}/\mathrm{d}\,\gamma) = 1$ is an artifact of managing the likelihood of loans being paid off early. This narrowly defined objective function clearly neglects aspects of expected profit. However, this analytic result nearly duplicates the simulated results for $(\mathrm{d}\,r^*_{\mathrm{L}}/\mathrm{d}\,\gamma)$ with T_{M} equal to 0.5 and then 1.0 years. In addition, it provides a basis for understanding the simulated values of $(\partial r^*_{\mathrm{L}}/\partial\gamma)$. An increase in the penalty rate of $\mathrm{d}\gamma$ allows the bank to increase the lending rate by $\mathrm{d}r_{\mathrm{L}}$ $(= \mathrm{d}\gamma)$ without changing the risk of disintermediation because the decrease in the benchmark rate necessary to make refinancing profitable is unchanged.

6 Conclusion

This paper has outlined a solution to the loan pricing problem faced by bankers in an environment of falling interest rates. We used the behavior of a benchmark rate to derive the probability density function of the time to prepayment. Then we determined the bank's inter-temporal expected profits and maximized them with respect to r_{L} to yield the optimal lending rate, r_{L}^*. A numerical solution for r_{L}^* was obtained. Since r_{L}^* is an implicit function of the loan's time to expiry we were able to determine the maturity structure of lending rates.

The comparative static behavior of r_{L}^* was then studied under a host of parametric schemes. An increase in σ shifted the optimal structure of lending rates downward while increases in either μ or γ shifted the maturity structure of r_{L}^* upward. The comparative static behavior of the optimal lending rate was easily understood if we thought in terms of the relationship between r_{L}^* and the likelihood of loan prepayment. Any parametric change that reduces expected profits invariably increases the risk of prepayment and occasions a reduction in r_{L}^*. If the parametric change increases expected profit, then the risk of prepayment falls so that the optimal lending rate increases.

References

Baker, M., Greenwood, R., Wurgler, J.: The maturity of debt issues and predictable variation in bond returns. J Financ Econ **70**, 261–291 (2003)

Cochrane, John H.: New facts in finance, federal reserve bank of chicago, economic perspectives **23**(3, Third-Qtr), 36–58 (1999)

Deng, Y., Quigley, J., Van Order, R.: Mortgage terminations, heterogeneity, and the exercise of mortgage options. Econometrica **68**(2), 275–307 (2000)

Dhillon, U.S., Lasser, D.J.: Term premium estimates from zero-coupon bonds: new evidence on the expectations hypothesis. J Fixed Income **8**(1, Jun), 52–58 (1998)

Gross, D.B., Souleles, N.S.: An empirical analysis of personal bankruptcy and delinquency. Rev Financ Stud **15**(1), 319–347 (2002)

Heitfield, E., Sabarwal, T.: What drives default and prepayment on subprime auto loans? J Real Estate Financ Econ **29**(4), 457–477 (2004)

Jones, C.M., Lamont, O., Lumsdaine, R.L.: Macroeconomic news and bond market volatility. J Financ Econ **47**(3, Mar), 315–337 (1998)

King, T.H.D.: An empirical examination of call option values implicit in US corporate bonds. JFQA **37**(4):693–721 (2002)

Mills, Terrence C.: The econometric modeling of financial time series. 2nd edn, Cambridge University Press, Cambridge, UK (1999)

Papageorgiou, Nicholas, Skinner, F.S.: Predicting the direction of interest rate movements. J Fixed Income **11**(4, Mar), 87–95 (2002)

Pennington-Cross, A.: Credit history and the performance of prime and nonprime mortgages. J Real Estate Financ Econ **27**(3), 279–301 (2003)

Pindyck, R.S., Rubinfeld, D.L.: Econometric models and economic forecasts. 4th edn, Irwin McGraw-Hill, Boston (1998)

Reisman, Haim, Zohar, G.: Short-term predictability of the term structure. J Fixed Income **14**(3, Dec), 7–14 (2004)

Shumway, T.: Forecasting bankruptcy more accurately: a simple hazard model. J Business **74**(1), 101–124 (2001)

Xu, Yexiao,: Small levels of predictability and large economic gains. J Empirical Financ **11**(2, Mar), 237–275 (2004)